SHARP BOUNDARY ESTIMATES FOR ELLIPTIC OPERATORS

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Abstract

We prove sharp L^2 boundary decay estimates for the eigenfunctions of certain second order elliptic operators acting in a bounded region, and of their first space derivatives, using only the Hardy inequality. These imply L^2 boundary decay properties of the heat kernel and spectral density. We deduce bounds on the rate of convergence of the eigenvalues when the region is slightly reduced in size. It is remarkable that several of the bounds do not involve the space dimension.

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1 Introduction

Let H be a non-negative second order elliptic operator acting in $L^2(U, \mathrm{d}^N x)$ subject to Dirichlet boundary conditions, where U is a bounded region in $\mathbf{R}^N$ or even in a Riemannian manifold. Let d be a continuous function on U satisfying $|\nabla d| \leq 1$, for example the distance from the boundary of U, which may be very irregular. We say that H satisfies Hardy's inequality with respect to d if

$$\int_U \frac{|f|^2}{d^2} \leq c^2 \left(Q(f) + a\|f\|^2 \right)$$

for all $f \in C_c^\infty(U)$, where Q is the quadratic form of H. The precise value of the constant c will be of great importance below, but the size of a is not crucial.

We are concerned with boundary decay of the eigenfunctions of H, and more generally of any functions in the domain of H. Such bounds were first obtained in [9, 6] by an argument related to that which we use below. The bounds were used in [6] to obtain explicit upper bounds on the rate at which

$$|\lambda_n(U) - \lambda_n(U_\varepsilon)|$$

vanishes as $\varepsilon \to 0$, where $\lambda_n(S)$ denotes the n-th Dirichlet eigenvalue of any region S and

$$U_\varepsilon := \{x \in U : d(x) > \varepsilon\}.$$

In two recent papers Pang [12, 13] used a different method to obtain a sharp rate of convergence of the first eigenvalue as $\varepsilon \to 0$ for a Dirichlet Laplacian in any simply connected subregion of $\mathbf{R}^2$, and for certain bounded regions in $\mathbf{R}^N$. In this paper we return to the method of [6] and show how to obtain sharp estimates of the rate of convergence for all eigenvalues; see Section 4.

The key is to obtain better boundary decay estimates of the eigenfunctions, in terms of

$$\int_{\{x:d(x)<\varepsilon\}} |f|^2$$

for all $f \in \mathrm{Dom}(H)$ and all $\varepsilon > 0$, instead of estimating

$$\int_U \frac{|f|^2}{d^\gamma}$$

for all possible $\gamma > 0$. It is well known to harmonic analysts that the former type of estimate is generally sharper than the latter, and we find that it yields the optimal power in the subsequent proof of the convergence of the eigenvalues.

The main theorems of the paper in Section 3 apply to weighted Laplace-Beltrami operators acting in regions with irregular and possibly fractal boundaries, but in Section 5 we show that the methods can be applied to second order uniformly elliptic operators with measurable highest order coefficients. The estimates are proved for functions in the domains of the operators, and apply in particular to eigenfunctions. In most theorems we prove that we have the optimal power of ε in the estimates.

The methods which we use do not require U to be a region in a Riemannian manifold. If U is a piecewise manifold obtained by glueing together manifolds of the same dimension along certain common edges, the same ideas can be applied provided the operator H is defined by means of the appropriate quadratic form.

In Section 6 we use the results to obtain some new L^2 boundary decay estimates for the heat kernel of the operator, and remark that the same methods can be used for the spectral density.

The sharp constant in Hardy's inequality is the only important input to the argument, and we refer to [8] for a recent review of this topic. Here we mention only a few outstanding results for $H := -\Delta_{DIR}$ acting in a bounded region U in Euclidean space. If U is a simply connected proper subregion of $\mathbf{R}^2$ then Hardy's inequality holds with $c = 4$ and $a = 0$ by [1], [5, Th. 1.5.10]. If U is a convex region in $\mathbf{R}^N$ then it is a matter of folklore that Hardy's inequality holds with $c = 2$ and $a = 0$; some proofs are described in [8]. Finally, if U has smooth boundary then Hardy's inequality holds with $c = 2$ for some $a < \infty$, [3].

2 Definitions

We follow the notation of [6] but with somewhat more restrictive conditions on the various coefficient functions. Let σ be a measurable function on the incomplete Riemannian manifold U which is positive almost everywhere and locally L^2 wth respect to the Riemannian volume element. Define the weighted space $L^2(U)$ to be the set of (equivalence classes modulo null sets of) functions such that

$$\|f\|_2^2 := \int_U |f|^2\sigma^2 < \infty.$$

This and subsequent integrals are evaluated using the Riemannian volume element. The introduction of the weight σ allows extra applications of our theorems at no cost. Let V be a non-negative locally L^1 function on U and let H be the operator on $L^2(U)$ defined formally by

$$Hf := -\sigma^{-2}\nabla\cdot\left(\sigma^2\nabla f\right) + Vf$$

subject to Dirichlet boundary conditions. Rigorously we start from the non-negative quadratic form

$$Q(f) := \int_U \left(|\nabla f|^2 + V|f|^2\right)\sigma^2$$

which is well-defined on the domain $C_c^\infty(U)$ by the hypothesis on σ. We assume that Q is closable and define H to be the self-adjoint operator on $L^2(U)$ associated with the closure of the form as described in [4, Ch. 4] and [5, Section 1.2]. For discussions of conditions on σ which imply that Q is closable see [14] and [5, Section 1.2].

If U is a region in $\mathbf{R}^N$, $\sigma = 1$, $V = 0$ and we choose the Euclidean metric then $H = -\Delta$ subject to Dirichlet boundary conditions. Our results are new in this case when U is bounded and its boundary ∂U is irregular, possibly fractal, improving on the recent theorems in [6, 12, 13].

Our main assumption is formulated in terms of a positive continuous function d on U such that $|\nabla d| \le 1$, in the weak sense. More precisely, we assume that

$$|d(x) - d(y)| \le |x-y|$$

for all $x, y \in U$. This is equivalent to the statement that d has distributional derivative $\nabla d \in L^\infty$ which satisfies $|\nabla d(x)| \le 1$ almost everywhere in U. One might take $d(x)$ to be the distance of $x \in U$ from any closed subset S of ∂U or from a closed subset of $M\backslash U$ if U is embedded in some larger Riemannian manifold M. We assume throughout the paper that for some constant $c \ge 2$ and some non-negative constant a the Hardy inequality (HI)

$$\int_U \frac{|f|^2}{d^2}\sigma^2 \le c^2\left(Q(f) + a\|f\|^2\right)$$

is valid for all $f \in C_c^\infty(U)$, and hence for all f in the domain of the closure of Q. Our goal is to obtain a similar but stronger bound for all $f \in \mathrm{Dom}(H)$ and hence for all eigenfunctions of H. Note that we do not assume that U is bounded or H has discrete spectrum.

A serious difficulty is the fact that we cannot identify the domain of H with any of the Sobolev or other spaces. If the coefficients of H or the boundary are irregular the operator domain of H changes if we vary σ or V within the permitted classes, even though the quadratic form domain may be unchanged. The bounds which we obtain in Theorems 4 and 7 bear some relationship with Morrey space estimates, already known to be of great importance in the theory of elliptic operators, [2, 10, 11].

3 The main theorems

Our estimates involve a positive parameter ε, and various other constants which depend only on $c \geq 2$, in a way which we make explicit. Given $\varepsilon > 0$ we put

$$\omega(x) := (\max\{d(x), \varepsilon\})^{-1/c}$$

for all $x \in U$.

Lemma 1 *If $f \in \mathrm{Dom}(H)$ and $s \geq 0$ then*

$$\left|\langle Hf, \omega^2 f\rangle + s\|\omega f\|_2^2\right| \leq c^{2/c}\|(H+s)f\|_2\|(H+a)^{1/c}f\|_2.$$

Proof Using HI and [4, Lemma 4.20] we have

$$\omega^4 \leq (d^{-2})^{2/c} \leq \{c^2(H+a)\}^{2/c}$$

so

$$0 \leq (H+a)^{-1/c}\omega^4(H+a)^{-1/c} \leq c^{4/c} I$$

and

$$\|\omega^2(H+a)^{-1/c}\| \leq c^{2/c}.$$

Hence

$$\begin{aligned}
&\left|\langle Hf, \omega^2 f\rangle + s\|\omega f\|_2^2\right| \\
&= \left|\langle (H+s)f, \omega^2 f\rangle\right| \\
&= \left|\langle (H+s)f, \omega^2(H+a)^{-1/c}.(H+a)^{1/c}f\rangle\right| \\
&\leq c^{2/c}\|(H+s)f\|_2\|(H+a)^{1/c}f\|_2.
\end{aligned}$$

Lemma 2 *If $f \in \mathrm{Dom}(Q)$ and $\mu \in W^{1,\infty}(U)$ then $\mu f \in \mathrm{Dom}(Q)$ and*

$$Q(\mu f) \leq 2\|\mu\|_\infty^2 Q(f) + 2\|\nabla\mu\|_\infty^2 \|f\|_2^2.$$

<u>Proof</u> If $f \in C_c^\infty(U)$ then $\mu f \in W_c^{1,\infty}(U) \subseteq \mathrm{Dom}(Q)$ and

$$\begin{aligned} Q(\mu f) &= \int_U \left(|\mu\nabla f + f\nabla\mu|^2 + V\mu^2|f|^2\right)\sigma^2 \\ &\leq \int_U \left(2\mu^2|\nabla f|^2 + 2|f|^2|\nabla\mu|^2 + V\mu^2|f|^2\right)\sigma^2 \\ &\leq 2\|\mu\|_\infty^2 Q(f) + 2\|\nabla\mu\|_\infty^2\|f\|_2^2. \end{aligned}$$

If $f \in \mathrm{Dom}(Q)$ then the fact that $\mu f \in \mathrm{Dom}(Q)$ and the validity of the same estimate both follow from the lower semi-continuity of Q.

Lemma 3 *If $f \in \mathrm{Dom}(H)$ then*

$$\int_U \frac{\omega^2|f|^2\sigma^2}{c^2d^2} \leq c^{2/c}\|(H+a)f\|_2\|(H+a)^{1/c}f\|_2 + \int_U |\nabla\omega|^2|f|^2\sigma^2.$$

<u>Proof</u> By HI and Lemma 2 we see that $\omega f \in \mathrm{Dom}(Q)$ and

$$\int_U \frac{\omega^2|f|^2\sigma^2}{c^2d^2} \leq Q(\omega f) + a\|\omega f\|_2^2.$$

Secondly

$$\begin{aligned} &Q(\omega f) - \frac{1}{2}\langle Hf, \omega^2 f\rangle - \frac{1}{2}\langle \omega^2 f, Hf\rangle \\ &= \int_U \left\{|\nabla(\omega f)|^2 - \frac{1}{2}\nabla f\cdot\overline{\nabla(\omega^2 f)} - \frac{1}{2}\nabla(\omega^2 f)\cdot\overline{\nabla f}\right\}\sigma^2 \\ &= \int_U |\nabla\omega|^2|f|^2\sigma^2. \end{aligned} \tag{1}$$

Hence

$$\begin{aligned} &Q(\omega f) + a\|\omega f\|_2^2 \\ &= \frac{1}{2}\left(\langle Hf, \omega^2 f\rangle + a\|\omega f\|_2^2\right) + \frac{1}{2}\left(\langle\omega^2 f, Hf\rangle + a\|\omega f\|_2^2\right) + \int_U |\nabla\omega|^2|f|^2\sigma^2. \end{aligned}$$

The proof is completed by combining the above two formulae with the bound of Lemma 1.

For some comments on the optimality of the estimates in the following theorem see Example 5, Example 6 and the note after Corollary 9.

Theorem 4 *If $f \in \mathrm{Dom}(H)$ then assuming HI we have*

$$\int_{\{x:d(x)<\varepsilon\}} \frac{|f|^2}{d^2}\sigma^2 \leq c_0\varepsilon^{2/c}\|(H+a)f\|_2\|(H+a)^{1/c}f\|_2$$

for all $\varepsilon > 0$*, where* $c_0 := c^{2+2/c}$*. Hence*

$$\int_{\{x:d(x)<\varepsilon\}} |f|^2\sigma^2 \leq c_0\varepsilon^{2+2/c}\|(H+a)f\|_2\|(H+a)^{1/c}f\|_2$$

for all $\varepsilon > 0$*.*

Proof We rewrite Lemma 3 in the form

$$\int_U Y|f|^2\sigma^2 \leq c^{2/c}\|(H+a)f\|_2\|(H+a)^{1/c}f\|_2$$

where

$$Y := \frac{\omega^2}{c^2d^2} - |\nabla\omega|^2.$$

If $d(x) \geq \varepsilon$ then

$$|\nabla\omega|^2 \leq c^{-2}d^{-2-2/c} = \frac{\omega^2}{c^2d^2}$$

so $Y(x) \geq 0$. On the other hand if $d(x) < \varepsilon$ then

$$Y = \frac{\omega^2}{c^2d^2} \geq \frac{1}{c^2\varepsilon^{2/c}d^2}.$$

Thus

$$\begin{aligned}\int_{\{x:d(x)<\varepsilon\}} \frac{|f|^2}{d^2}\sigma^2 &\leq c^2\varepsilon^{2/c}\int_U Y|f|^2\sigma^2 \\ &\leq c^{2+2/c}\varepsilon^{2/c}\|(H+a)f\|_2\|(H+a)^{1/c}f\|_2.\end{aligned}$$

The second statement of the theorem is an immediate consequence of the first.

Example 5 If $c = 2$ and $\sigma = 1$ then the theorem states that

$$\int_{\{x:d(x)<\varepsilon\}} |f|^2 \leq 8\varepsilon^3\|(H+a)f\|_2\|(H+a)^{1/2}f\|_2$$

for all $f \in \mathrm{Dom}(H)$, which is the L^2 analogue of $f(x) = O(d(x))$ as $d(x) \to 0$. In particular suppose that $U \subseteq \mathbf{R}^N$ is bounded with a smooth boundary ∂U, and let f be a generic function in $C_c^\infty(\overline{U})$ which vanishes on ∂U. Then $f \in \mathrm{Dom}(H)$ and the power 3 of ε above is optimal.

Example 6 Let $U := (0,\infty)$, $d(x) := x$ and $\sigma(x) := x^{\alpha/2}$ where $0 \leq \alpha < 1$. Then the operator H is given formally by

$$Hf(x) := -x^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}x}\left\{x^\alpha\frac{\mathrm{d}f}{\mathrm{d}x}\right\}$$

subject to Dirichlet boundary conditions at $x = 0$. The quadratic form Q has domain $W_0^{1,2}((0,\infty), x^\alpha\mathrm{d}x)$. A standard result, [7, p. 104], states that the strong Hardy inequality holds with $c = 2/(1-\alpha)$.

Now let f be a smooth function on $(0,\infty)$ which vanishes for $x>2$ and equals $x^{1-\alpha}$ for $0<\alpha<1$. It is easy to prove that $f\in\mathrm{Dom}(H)\subseteq W_0^{1,2}((0,\infty),x^\alpha\mathrm{d}x)$. If $0<\varepsilon<1$ then one also has

$$\int_{\{x:d(x)<\varepsilon\}}\frac{|f|^2}{d^2}\sigma^2=\frac{\varepsilon^{3-\alpha}}{3-\alpha}=k\varepsilon^{2+2/c}.$$

Therefore the power $2+2/c$ in Theorem 4 is optimal.

Corollary 7 *If $g(s)$ is a monotonically decreasing C^1 function on $(0,\delta]$ which vanishes for $s=\delta$ then*

$$\int_U g(d)|f|^2\sigma^2\leq c_0\int_0^\delta |g'(s)|s^{2+2/c}\mathrm{d}s.\|(H+a)f\|_2\|(H+a)^{1/c}f\|_2 \tag{2}$$

for all $f\in\mathrm{Dom}(H)$, provided the integral on the RHS is finite.

Proof We have

$$g(d)=\int_d^\delta |g'(s)|\mathrm{d}s$$

for all $d\in(0,\delta]$. The corollary follows by applying Fubini's theorem to

$$\int_{s=0}^\delta\left\{\int_U \chi_{d<s}|g'(s)|\,|f|^2\sigma^2\right\}\mathrm{d}s$$

where χ stands for the characteristic function of a set.

Note Let $H:=-\Delta_{DIR}$ acting in $L^2(U,\mathrm{d}x)$ with $\sigma:=1$, where U is a bounded region in $\mathbf{R}^N$, and let d be the distance to the boundary ∂U. If $g(s)=o(s^{-2-2/c})$ as $s\to 0$ then (2) is equivalent to

$$\int_U g(d)|f|^2\leq c_0(2+2/c)\int_0^\delta g(s)s^{1+2/c}\mathrm{d}s.\|(H+a)f\|_2\|(H+a)^{1/c}f\|_2 \tag{3}$$

which may be compared with the pointwise bound

$$|f(x)|=O(d(x)^{1/2+1/c}) \tag{4}$$

as $d(x)\to 0$. If ∂U is C^2 then $c=2$ and this pointwise bound is sharp for the first eigenfunction of H. However, no such pointwise bound exists for arbitrary functions in the domain of H. Moreover, if ∂U is fractal it is not clear that (4) is the correct pointwise analogue of (3), nor indeed that there is any pointwise analogue.

Our next task is to obtain comparable estimates for $|\nabla f|$. This necessitates introducing the continuous function on $\tau:U\to[0,\infty)$ defined by

$$\tau(x):=\begin{cases}\varepsilon^{-1/c} & \text{if } 0<d(x)\leq\varepsilon\\ c^{-1}\varepsilon^{-1-1/c}((1+c)\varepsilon-d(x)) & \text{if } \varepsilon<d(x)\leq(1+c)\varepsilon\\ 0 & \text{otherwise.}\end{cases}$$

It is immediate that

$$\begin{aligned} 0 \le \tau &\le \omega \le \varepsilon^{-1/c} \\ |\nabla \tau| &\le c^{-1}\varepsilon^{-1-1/c} \\ \mathrm{supp}(\tau) &\subseteq \{x : 0 \le d \le (1+c)\varepsilon\}. \end{aligned}$$

Theorem 8 *If $f \in \mathrm{Dom}(H)$ then assuming HI we have*

$$\int_{\{x:d(x)<\varepsilon\}} |\nabla f|^2 \sigma^2 \le c_1 \varepsilon^{2/c} \|(H+a)f\|_2 \|(H+a)^{1/c} f\|_2.$$

for all $\varepsilon > 0$, where

$$c_1 := c^{2/c} + c^{2/c}(1+c)^{2+2/c}.$$

<u>Proof</u> We have

$$\begin{aligned} &\varepsilon^{-2/c} \int_{\{x:d(x)<\varepsilon\}} |\nabla f|^2 \sigma^2 \\ &= \int_{\{x:d(x)<\varepsilon\}} |\nabla(\tau f)|^2 \sigma^2 \\ &\le Q(\tau f) \end{aligned}$$

where $\tau f \in \mathrm{Dom}(Q)$ by Lemma 2. By the same argument as in (1) of Lemma 3 this equals

$$\begin{aligned} &\frac{1}{2}\langle Hf, \tau^2 f\rangle + \frac{1}{2}\langle \tau^2 f, Hf\rangle + \int_U |\nabla \tau|^2 |f|^2 \sigma^2 \\ &\le \|Hf\|_2 \|\tau^2 (H+a)^{-1/c}\| \, \|(H+a)^{1/c} f\|_2 + \int_U |\nabla \tau|^2 |f|^2 \sigma^2 \\ &\le c^{2/c} \|Hf\|_2 \|(H+a)^{1/c} f\|_2 + c^{-2}\varepsilon^{-2-2/c} \int_{\{x:d(x)<(1+c)\varepsilon\}} |f|^2 \sigma^2 \\ &\le c_1 \|(H+a)f\|_2 \|(H+a)^{1/c} f\|_2 \end{aligned}$$

using Theorem 4.

Note By extending the calculation of Example 9 one sees that the power of ε in the above theorem is optimal. The choice of τ in the proof is certainly not optimal, so neither is the value of c_1 obtained.

Corollary 9 *If $Hf = \lambda f$ and $\|f\|_2 = 1$ then*

$$\int_{\{x:d(x)<\varepsilon\}} |f|^2 \sigma^2 \le c_0 \varepsilon^{2+2/c} (\lambda + a)^{1+1/c} \tag{5}$$

and

$$\int_{\{x:d(x)<\varepsilon\}} |\nabla f|^2 \sigma^2 \le c_1 \varepsilon^{2/c} (\lambda + a)^{1+1/c}$$

for all $\varepsilon > 0$.

Proof These follow directly from Theorems 4 and 6.

Note If we insert the eigenfunction f directly into HI we obtain

$$\int_{\{x:d(x)<\varepsilon\}} |f|^2\sigma^2 \leq c^2\varepsilon^2(\lambda+a)$$

which is exactly what is obtained by interpolating between (5) and the trivial estimate

$$\int_{\{x:d(x)<\varepsilon\}} |f|^2\sigma^2 \leq 1.$$

This supports the conjecture that the constant c_0 in Theorem 4 is optimal.

Corollary 10 *If $H := -\Delta_{DIR}$ in $L^2(U, \mathrm{d}^2x)$ where U is a simply connected proper subregion of $\mathbf{R}^2$ and*

$$d(x) := \mathrm{dist}(x, \partial U)$$

then

$$\int_{\{x:d(x)<\varepsilon\}} |f|^2 \leq 32\varepsilon^{5/2}\|Hf\|_2\|H^{1/4}f\|_2$$

and

$$\int_{\{x:d(x)<\varepsilon\}} |\nabla f|^2 \leq 114\varepsilon^{1/2}\|Hf\|_2\|H^{1/4}f\|_2$$

for all $f \in \mathrm{Dom}(H)$.

Proof We may put $c = 4$, $a = 0$ and $\sigma = 1$ in Theorems 4 and 6 by [1], [5, Th. 1.5.10].

4 Perturbation of the domain

In this section we use the results above to consider the effect on the spectrum of H of replacing the region U by a slightly smaller region U_ε such that

$$\{x \in U : d(x) > \varepsilon\} \subseteq U_\varepsilon \subseteq U.$$

If $\lambda_n(U_\varepsilon)$ denote the eigenvalues of the operator H_ε defined by restricting H to $L^2(U_\varepsilon)$ where we again impose Dirichlet boundary conditions, then variational arguments imply that

$$\lambda_n(U) \leq \lambda_n(U_\varepsilon)$$

for all n and $\varepsilon > 0$, and our goal is to find quantitative estimates of the difference. The constants c_n below all depend only on a, c, c_0, c_1 and n.

Let $\mu : U \to [0, \infty)$ be defined by

$$\mu(x) := \begin{cases} 0 & \text{if } 0 < d(x) \leq \varepsilon \\ (d(x)-\varepsilon)/\varepsilon & \text{if } \varepsilon < d(x) \leq 2\varepsilon \\ 1 & \text{otherwise.} \end{cases}$$

so that $0 \leq \mu \leq 1$, $|\nabla\mu| \leq \varepsilon^{-1}$ and μ has support in U_ε.

Lemma 11 *There exists a constant $c_2 \geq 0$ such that if $f \in \mathrm{Dom}(H)$ then*

$$Q(\mu f) \leq Q(f) + \varepsilon^{2/c} c_2 \|(H+a)f\|_2 \|(H+a)^{1/c} f\|_2.$$

Proof Putting $S := \{x : \varepsilon < d(x) < 2\varepsilon\}$ we have

$$\begin{aligned} Q(\mu f) - Q(f) &\leq \int_S |\nabla(\mu f)|^2 \sigma^2 \\ &\leq 2\int_S \mu^2 |\nabla f|^2 \sigma^2 + 2\int_S |\nabla \mu|^2 |f|^2 \sigma^2 \\ &\leq 2\int_S |\nabla f|^2 \sigma^2 + 2\varepsilon^{-2} \int_S |f|^2 \sigma^2 \\ &\leq \varepsilon^{2/c} c_2 \|(H+a)f\|_2 \|(H+a)^{1/c} f\|_2 \end{aligned}$$

by Theorems 4 and 6.

It is crucial to the application of our next lemma that $0 < \varepsilon^{1+1/c} < \varepsilon^{2/c}$ provided $0 < \varepsilon < 1$, so the error is actually smaller than that of Lemma 9 as $\varepsilon \to 0$.

Lemma 12 *There exists a constant $c_3 \geq 0$ such that if $f \in \mathrm{Dom}(H)$ then*

$$\|f\|_2 \geq \|\mu f\|_2 \geq \|f\|_2 - c_3 \varepsilon^{1+1/c} \|(H+a)f\|_2^{1/2} \|(H+a)^{1/c} f\|_2^{1/2}.$$

Proof The first inequality is elementary. We also have

$$\begin{aligned} |\|f\|_2 - \|\mu f\|_2|^2 &\leq \|f - \mu f\|_2^2 \\ &= \int_U (1-\mu)^2 |f|^2 \sigma^2 \\ &\leq \int_{\{x : d(x) < 2\varepsilon\}} |f|^2 \sigma^2 \\ &= \varepsilon^{2+2/c} c_3^2 \|(H+a)f\|_2 \|(H+a)^{1/c} f\|_2 \end{aligned}$$

by Theorem 4. The second inequality of the lemma follows.

The case $n = 1$ of the following theorem with the sharp power $\varepsilon^{1/2}$ corresponding to $c = 4$ was already proved for proper simply connected subregions of $\mathbf{R}^2$ in [13], by an entirely different method which seems not to extend to higher eigenvalues.

Theorem 13 *There exist constants c_n for all positive integers n such that*

$$\lambda_n(U) \leq \lambda_n(U_\varepsilon) \leq \lambda_n(U) + c_n \varepsilon^{2/c}.$$

Proof this follows [6, Th. 22] closely.

5 Elliptic operators

In this section we extend the earlier results to second order uniformly elliptic operators in divergence form with possibly measurable second order coefficients,

making use only of the ellipticity constant of the operator. Throughout the section we put $\sigma = 1$ and integrate with respect to Lebesgue measure.

Let U be a bounded region in $\mathbf{R}^N$ with C^2 boundary and let

$$d(x) := \mathrm{dist}(x, \partial U)$$

so that

$$\int_U \frac{|f|^2}{d^2} \leq 4 \int_U (|\nabla f|^2 + a|f|^2)$$

for some $a \geq 0$ and all $f \in W_0^{1,2}(U)$. Now let

$$Hf(x) := -\sum_{i,j} \frac{\partial}{\partial x_i} \left(a^{i,j}(x) \frac{\partial f}{\partial x_j} \right)$$

subject to Dirichlet boundary conditions in $L^2(U)$, where

$$1 \leq a(x) \leq \alpha^2$$

for all $x \in U$, and we interpret H as a self-adjoint operator using the theory of quadratic forms as usual. If we put $\tilde{d}(x) := \alpha^{-1} d(x)$ then

$$\sum_{i,j} a^{i,j}(x) \frac{\partial \tilde{d}}{\partial x_i} \frac{\partial \tilde{d}}{\partial x_j} \leq 1$$

for all $x \in U$ and

$$\int_U \frac{|f|^2}{\tilde{d}^2} \leq 4\alpha^2 \left(Q(f) + a\|f\|_2^2 \right)$$

for all $f \in W_0^{1,2}(U)$, where Q is the quadratic form asociated with H.

We are now in a position to apply the theory of the paper to the pair $H, \tilde{d}$ with $c := 2\alpha$.

Theorem 14 *There exists a constant c_0 such that if $f \in \mathrm{Dom}(H)$ then*

$$\int_{\{x: d(x) < \varepsilon\}} |f|^2 \leq c_0 \varepsilon^{2+1/\alpha} \|(H+a)f\|_2 \|(H+a)^{1/(2\alpha)} f\|_2. \tag{6}$$

Theorem 15 *There exists a constant c_1 such that if $f \in \mathrm{Dom}(H)$ then*

$$\int_{\{x: d(x) < \varepsilon\}} |\nabla f|^2 \leq c_1 \varepsilon^{1/\alpha} \|(H+a)f\|_2 \|(H+a)^{1/(2\alpha)} f\|_2. \tag{7}$$

We next suppose that U_ε is a region satisfying the same conditions as in Theorem 11, and define $\lambda_n(U_\varepsilon)$ in a similar manner.

Theorem 16 *There exist constants c_n for all positive integers n such that*

$$\lambda_n(U) \leq \lambda_n(U_\varepsilon) \leq \lambda_n(U) + c_n \varepsilon^{1/\alpha}.$$

In each case we conjecture that the power of ε is optimal. The three theorems can be proved in two ways. We may adapt the proofs of this paper, replacing the weighted Laplacian by a more general second order elliptic operator. Alternatively, we may apply the theorems of the paper, but using a Riemannian metric and weight adapted to the choice of the second order coefficients, as described in [6]. Namely if $g_{i,j}(x)$ is the matrix inverse to $a^{i,j}(x)$ then the Riemannian metric

$$\sum_{i,j} g_{i,j}(x)\mathrm{d}x^i\mathrm{d}x^j$$

is Lipschitz equivalent to the Euclidean metric in U. Indeed the Riemannian distance function is bounded between $\alpha^{-1}d$ and d. If also

$$\sigma(x) := \det\left(g_{i,j}(x)\right)^{-1/4}$$

then

$$\begin{aligned}\int_U |\nabla f|^2\sigma^2 &= \int_U \sum_{i,j} a^{i,j}(x)\frac{\partial f}{\partial x_i}\frac{\partial \overline{f}}{\partial x_j}\mathrm{d}^N x\\ \int_U |f|^2\sigma^2 &= \int_U |f|^2\mathrm{d}^N x\end{aligned}$$

where the integrals on the left are with respect to the Riemannian measure and $\mathrm{d}^N x$ is the Lebesgue measure. Hence

$$\begin{aligned}Hf &:= -\sigma^2\nabla\cdot\left(\sigma^2\nabla f\right)\\ &= -\sum_{i,j}\frac{\partial}{\partial x_i}\left(a^{i,j}(x)\frac{\partial f}{\partial x_j}\right).\end{aligned}$$

We finally remark that the above theorems can be localised. Suppose H is subelliptic but the hypothesis $1 \le a(x) \le \alpha^2$ holds for x such that $\mathrm{dist}(x, S) < \beta$, where $\beta > 0$ and S is some closed subset of ∂U. We only assume that ∂U is C^2 in the β-neighbourhood of S. If we put

$$\tilde{d}(x) = \alpha^{-1}\min\{\mathrm{dist}(x, S), \beta\}$$

then

$$\sum_{i,j} a_{i,j}(x)\frac{\partial\tilde{d}}{\partial x_i}\frac{\partial\tilde{d}}{\partial x_j} \le 1$$

for all $x \in U$, because the gradient of $\tilde{d}$ vanishes outside the β-neighbourhood of S. The proof of

$$\int_U \frac{|f|^2}{\tilde{d}^2} \le 4\alpha^2\left(Q(f) + a\|f\|_2^2\right)$$

for all $f \in W_0^{1,2}(U)$ involves the same arguments as in [3], concentrating on the region $\{x \in U : \mathrm{dist}(x, S) < \beta\}$.

6 Heat kernel and related bounds

If $K(t,x,y)$ is the heat kernel of a uniformly elliptic second order operator H written in divergence form and acting in $L^2(U,\mathrm{d}^N x)$ subject to Dirichlet boundary conditions, then it is known that

$$0 \le K(t,x,y) \le c_2 t^{-N/2} \tag{8}$$

for all $x,y\in U$ and all $t>0$. If $H=-\Delta_{DIR}$ then we may even take $c_2=(4\pi)^{-N/2}$; see [5] for an account of the relevant heat kernel bounds. We are interested in L^2 boundary decay properties of the heat kernel and spectral density which bear some relationship with the 'intrinsically ultracontractive' pointwise bounds obtained under much stronger assumptions and with much less control on the constants in [5, Chapter 4]. Throughout this section we assume (6), (7) and (8); the constants in our bounds depend on these constants and on N in a manner which is easy to make explicit.

Theorem 17 *Under the assumptions (6), (7) and (8) we have*

$$\int_{\{x:d(x)<\varepsilon\}} K(t,x,y)^2 \mathrm{d}^N x \le c_3 \mathrm{e}^{at}(\varepsilon^2/t)^{1+1/(2\alpha)} t^{-N/2} \tag{9}$$

for all $y\in U$ and all $t>0$. If also U is bounded then

$$\int_{\{x:d(x)<\varepsilon\}} K(t,x,x) \mathrm{d}^N x \le c_4 (\varepsilon^2/t)^{1+1/(2\alpha)} t^{-N/2} \tag{10}$$

for all $\varepsilon>0$ and all $0<t\le 1$.

Proof Denoting the left-hand side of (9) by I we have

$$I=\int_{\{x:d(x)<\varepsilon\}} |\mathrm{e}^{-Ht}\delta_y|^2 \mathrm{d}^N x = \int_{\{x:d(x)<\varepsilon\}} |\mathrm{e}^{-Ht/2} g|^2 \mathrm{d}^N x$$

where δ_y is the delta function at $y\in U$ and $g:=\mathrm{e}^{-Ht/2}\delta_y$. Using the semigroup property we have

$$\|g\|_2^2 = \int_U K(t/2,x,y)^2 \mathrm{d}^N x = K(t,y,y) \le c_2 t^{-N/2}.$$

Applying (6) and then the spectral theorem we deduce

$$\begin{aligned} I &\le c_0 \varepsilon^{2+1/\alpha} \|(H+a)\mathrm{e}^{-Ht/2} g\|_2 \|(H+a)^{1/(2\alpha)} \mathrm{e}^{-Ht/2} g\|_2 \\ &\le c_0 \varepsilon^{2+1/\alpha} \mathrm{e}^{at} \|(H+a)\mathrm{e}^{-(H+a)t/2}\| \, \|(H+a)^{1/(2\alpha)} \mathrm{e}^{-(H+a)t/2}\| \, \|g\|_2^2 \\ &\le c_3 \varepsilon^{2+1/\alpha} \mathrm{e}^{at} t^{-1-1/(2\alpha)-N/2}. \end{aligned}$$

We adopt an alternative strategy to prove the second inequality. Let $\{\lambda_n\}_{n=1}^\infty$ be the eigenvalues of H written in increasing order and repeated according to multiplicity,

and let $\{\phi_n\}_{n=1}^{\infty}$ be the corresponding normalised eigenfunctions. It is known that there exist positive constants a_1 and a_2 such that

$$a_1 n^{2/N} \leq \lambda_n \leq a_2 n^{2/N}$$

for all n. Also

$$K(t,x,x) = \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t} |\phi_n(x)|^2$$

for all $x \in U$ and $t > 0$. Denoting the left-hand side of (10) by J we deduce that for $0 < t \leq 1$

$$\begin{aligned} J &= \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t} \int_{\{x:d(x)<\varepsilon\}} |\phi_n|^2 \mathrm{d}^N x \\ &\leq \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t} c_0 \varepsilon^{2+1/\alpha} (\lambda_n + a)^{1+1/(2\alpha)} \\ &\leq c_0 \varepsilon^{2+1/\alpha} \sum_{n=1}^{\infty} \mathrm{e}^{-a_1 n^{2/N} t} (a_2 n^{2/N} + a)^{1+1/(2\alpha)} \\ &\leq c_4 \varepsilon^{2+1/\alpha} t^{-1-1/(2\alpha)-N/2} \end{aligned}$$

where for the last line we compared the sum with the corresponding integral.

The estimate (10) is not asymptotically optimal as $\varepsilon, t \to 0$, even for $H := -\Delta_{DIR}$ acting in $L^2(0,\infty)$. In this case we have

$$\begin{aligned} K(t,x,x) &= (4\pi t)^{-1/2} \left(1 - \mathrm{e}^{-x^2/t}\right) \\ &\sim (4\pi)^{-1/2} x^2 t^{-3/2} \end{aligned}$$

if $0 < x^2 << t$ by the reflection principle [5, p107]. Therefore

$$\int_0^{\varepsilon} K(t,x,x) \mathrm{d}x \sim (36\pi)^{-1/2} \varepsilon^3 t^{-3/2}$$

if $0 < \varepsilon^2 << t$. However Theorem 15 with $\alpha = 1$ only yields

$$\int_0^{\varepsilon} K(t,x,x) \mathrm{d}x \leq c_4 \varepsilon^3 t^{-2}.$$

The following is a possible reason for this failure. Optimal estimates on eigenfunctions associated with highly degenerate eigenvalues can be much worse than one expects for typical eigenvalues. Our proof uses a bound for every eigenfunction which takes no account of this fact, so when summed up it is not surprising that the resulting heat kernel bound is not optimal.

We note that the same method may be used to obtain upper bounds on

$$\int_{\{x:d(x)<\varepsilon\}} \sum_{n=1}^{\infty} \mathrm{e}^{-\lambda_n t} |\nabla \phi_n(x)|^2 \mathrm{d}^N x.$$

Another approach may be used to obtain upper bounds on quantities associated with the spectral density. Let E_λ be the spectral projection of H associated with the interval $(-\infty, \lambda)$, and let $e(\lambda, x, y)$ be its integral kernel. Then

$$N(\lambda) = \int_U e(\lambda, x, x)\mathrm{d}x$$

where $N(\lambda)$ is the number of eigenvalues of H less than λ. If $\varepsilon > 0$ we put

$$N(\varepsilon, \lambda) := \int_{\{x: d(x) < \varepsilon\}} e(\lambda, x, x)\mathrm{d}x.$$

Theorem 18 *Under the assumption (6) we have*

$$N(\varepsilon, \lambda) \le c_0 \varepsilon^{2+1/\alpha}(\lambda + a)^{1+1/(2\alpha)} N(\lambda)$$

for all $\lambda \ge 0$ *and* $\varepsilon > 0$.

Proof If $f = E_\lambda f$ then (6) implies

$$\int_{\{x: d(x) < \varepsilon\}} |f|^2 \le c_0 \varepsilon^{2+1/\alpha}(\lambda + a)^{1+1/(2\alpha)} \|f\|_2^2$$

which is equivalent to the operator inequality

$$\|Q_\varepsilon E_\lambda\|^2 \le c_0 \varepsilon^{2+1/\alpha}(\lambda + a)^{1+1/(2\alpha)}$$

where Q_ε is the projection given by multiplying by the characteristic function of $\{x : d(x) < \varepsilon\}$. We have

$$\begin{aligned} N(\varepsilon, \lambda) &= \mathrm{tr}[Q_\varepsilon E_\lambda Q_\varepsilon] \\ &= \mathrm{tr}[(Q_\varepsilon E_\lambda) E_\lambda (E_\lambda Q_\varepsilon)] \\ &\le \|Q_\varepsilon E_\lambda\|^2 \mathrm{tr}[E_\lambda] \\ &\le c_0 \varepsilon^{2+1/\alpha}(\lambda + a)^{1+1/(2\alpha)} N(\lambda). \end{aligned}$$

We finally comment that lower bounds on integrals associated with the spectral density have recently been obtained by Safarov, using a coherent state method, [15].

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